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ANALYTICAL SOLUTION FOR HEAT CONDUCTION IN ANISOTROPIC MEDIA IN INFINITE, SEMI-INFINITE, AND TWO-PLANE-BOUNDED REGIONS

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Abstract—This paper is one of a series to be reported in open literature concerning the analytical solution for heat conduction in anisotropic media. According to the mathematical difficulties and methods of solution, problems are divided into three classes. In this paper, only problems of the first class with anisotropy homogeneous in rectangular coordinates are solved through the use of Green's functions. The solution of a specific problem is shown and effects of material anisotropy to the temperature field and heat flow are discussed from both mathematical and physical points of view.

NOMENCLATURE

b, 1 or 0 defined in (2.3);
c, specific heat;
f, boundary data, defined in (2.3);
g, function defined in (4.4);
G, Green's function;
h^*, heat-transfer coefficient;
h, h^*/k_{22};
k_{ij}, thermal conductivity coefficients;
K_0, modified Bessel function of second kind and order zero;
L, spacing between two parallel planes;
Q, rate of heat production per unit volume;
R, pseudo-geodesic distance;
S, surface;
T, temperature;
x_i, x, y, z rectangular coordinates.

Greek symbols

α_2, $k_{22}/(\rho c)$;
β_{11}, $(\nu_{11}-\nu_{12}^2)^{1/2}$;
β_{33}, $(\nu_{33}-\nu_{23}^2)^{1/2}$;
ν_{ij}, k_{ij}/k_{22};
γ, $\nu_{13}-\nu_{12}\nu_{23}$;
σ, $(\beta_{33}^2-\gamma^2/\beta_{11}^2)^{1/2}$;
ε, $\nu_{23}-\nu_{12}\gamma/\beta_{11}^2$;
ξ, reduced coordinates defined by (8.4).

Subscripts

i, j, 1, 2, 3;
s, surface.

1. INTRODUCTION

IN RECENT years, the study on anisotropic materials has been of great interest in applied science and engineering, because of the rapidly increasing use of laminated structures, crystals and heat shielding materials. Many natural substances such as woods and sedimentary rocks are anisotropic. Metals which have undergone heavy cold processing also exhibit some kinds of anisotropy. In spite of the importance of anisotropic problems, reported results of their analytical solutions have been mostly done in crystal physics [1, 2] and elasticity [3–6].

Reported results of analytical solution of anisotropic problems with heat conduction as the principal subject have been very limited. Turkan and Tuna [7] extended an "approximate continuum theory on elasticity" to the solution of heat conduction in infinite-composite slabs and cylinders; but no result for the anisotropic case was reported. An exact solution of the same problem was reported by Padovan [8] in the form of triple series with discrete eigenvalues in all three directions. However, it may be pointed out that the exact solution for infinite composites of even isotropic media has not been reported to date, because of the difficulty in the calculation of eigenvalues with respect to the coordinate normal to the laminates. This difficulty is due to the fact that, if the laminates extend to infinity in one direction, then the eigenvalues in that direction are to be of continuous spectrum. This difficulty can be easily conceived from a recent paper of Horvay *et al.* [9].

Numerical solution of anisotropic problems has become possible since the advent of electronic computers. Katayama [10] employed the finite difference method and found his calculated results in good agreement with his experimental data. Chang *et al.* [11] employed the integral-equation method to calculate the temperature and heat flux distributions in a square, a circular disk, and an annular disk. Chen [12] applied the same method for the solution of anisotropic heat conduction in arbitrarily shaped domains. Cobble [13] solved the heat conduction in a wedge by first transforming the partial differential equation into an ordinary differential equation and then solving it numerically. The above studies were made on boundary conditions of Dirichlet type. For other boundary conditions, the calculations will be more complicated, especially for steady-state problems in open domains.

Experimental determination of thermal conductivity coefficients of general anisotropic materials may need the three-dimensional solution for a given specimen, for the simplification of two-dimensional analysis is useful only if material body possesses a plane of material symmetry [4].

According to our study on anisotropic problems of heat conduction, we find that the analytical solution of an anisotropic problem is in general difficult to obtain, especially in a finite region, but the Green's function can be constructed with less difficulty and for most cases can be expressed in terms of tabulated functions. We have also found that it is more convenient to divide anisotropic problems into three classes [14]. The first class considers the region bounded by not more than two surfaces normal to one spatial coordinate. The second class considers the region bounded or partially bounded by more than two surfaces on which the boundary conditions of the mixed type are limited on two parallel surfaces. The third class considers the region with boundary conditions of the mixed type on more than two surfaces. We shall report our studied results according to the above sequence.

In this paper, only problems of heat conduction in free space, half space and a slab are analyzed. These problems are clearly of the first class. Both steady and unsteady states with boundary conditions of Dirichlet, Neumann and mixed types are considered. For brevity, these boundary conditions will be referred to as the first, second, and third kinds, respectively. We shall concern mainly with the construction of Green's functions, since once the Green's function is known, the solution of a problem may be considered as complete, just as in the solution of isotropic problems [15–17]. An example will be shown to facilitate the discussion on anisotropic effects.

Several methods for the solution of the first-class problems may be used, such as the use of complex variables for two-dimensional problems [5, 6], the separation of variables [18], and integral transforms [19]. The latter method will be used in this paper in order that Green's functions for all the problems can be systematically presented. From results thus obtained, a transformation of coordinates is discovered.

2. FUNDAMENTAL EQUATIONS AND FORMAL SOLUTIONS

Consider an anisotropic medium which is homogeneous in rectangular coordinates* and has constant thermo-physical properties. The differential equations to be solved for unsteady and steady states are respectively [15]

$$\rho c \frac{\partial T}{\partial t} - k_{ij} \frac{\partial^2 T}{\partial x_i \partial x_j} - Q(x_i, t) = 0 \qquad \text{in } \Omega \text{ for } t > 0 \tag{2.1}$$

$$k_{ij} \frac{\partial^2 T}{\partial x_i \partial x_j} + Q(x_i) = 0 \qquad \text{in } \Omega \tag{2.2}$$

where the summation convention has been followed; Ω is the region concerned, and other quantities have been defined in the Nomenclature; coefficients k_{ij} are assumed symmetrical, i.e. $k_{ij} = k_{ji}$ for $i \neq j$; and according to irreversible thermodynamics, $k_{ii} > 0$; $k_{ii}k_{jj} - k_{ij}^2 > 0$ for $i \neq j$; and k_{ij} can be either positive or negative [20].

The boundary conditions on T may be written in the general form

$$b \frac{\partial T}{\partial n^+} + hT = f \qquad \text{on } S \text{ for } t > 0 \tag{2.3}$$

$$T = F \qquad \text{in } \Omega \text{ for } t = 0 \tag{2.4}$$

where S is the surface of the region concerned; b is a pure number, and either h or b may be zero or unity so that boundary conditions of other kinds are included; f, F and Q are known functions and assumed to satisfy Hölder conditions or be square integrable;† for unsteady problems, f may also depend on time; and n^+ is the conormal so that the transverse derivative is

$$\frac{\partial}{\partial n^+} = \pm v_{ij} \frac{\partial}{\partial x_j} \tag{2.5}$$

where the plus and minus signs are for surfaces at $x_i > 0$ and $x_i = 0$, respectively. For steady problems, condition (2.4) drops and condition (2.3) remains in the same form, but f depends on spatial coordinates only.

If $G(x_i, t | x_i', t')$ denotes the Green's function associated with problem (2.1), (2.3), (2.4), then with $b = 1$, or $b = 0$ and $h = 1$, the temperature can be obtained by the use of Green's second formula [21].

$$T(x_i, t) = \int_\Omega F(x_i') G(x_i, t | x_i', 0) \, d\Omega(x_i') + \frac{1}{\rho c} \int_0^t \int_\Omega Q(x_i', t') G(x_i, t | x_i', t') \, d\Omega(x_i') \, dt'$$
$$- \alpha_2 \int_0^t \int_S f(x_s', t') \frac{\partial}{\partial n^+} G(x_i, t | x_s', t') \, dS(x_s') \, dt'. \tag{2.6}$$

*An anisotropic material which is homogeneous in one coordinate system becomes heterogeneous in other coordinate systems.

†For two-dimensional steady problems in free and half spaces, f and Q are more restrictive [5, 17].

If surface conditions are of Neumann's type, i.e. $h = 0$ and $b = 1$ in (2.3), then the last integral in (2.6) is replaced by

$$-\alpha_2 \int_0^t \int_S f(x_s', t')G(x_i, t|x_s', t')\,dS(x_s')\,dt'. \tag{2.7}$$

For steady problems, we just drop the first integral in (2.6) and remove the time variable, integrals with respect to time, and the factor α_2, and change the factor $1/\rho c$ in (2.6) by $1/k_{22}$ [22].

3. DETERMINATION OF GREEN'S FUNCTIONS

In the following, we shall use x, y, z to designate the spatial coordinates while x_i will be used only to represent a point in three-dimensional space. We let the semi-infinite region be over the plane $y = 0$ and the slab be bounded by planes $y = 0$ and $y = L$.

Consider first the unsteady problem. The Green's function is to satisfy [23].

$$v_{ij}\frac{\partial^2 G}{\partial x_i \partial x_j} - \frac{1}{\alpha_2}\frac{\partial G}{\partial t} = -\frac{1}{\alpha_2 w(x_i)}\delta(x_i - x_i')\delta(t - t')$$

and homogeneous initial and boundary conditions, where $w(x_i)$ is a weight function yet to be determined. For $-\infty < x < \infty$ and $-\infty < z < \infty$, we have the Fourier transform of $G(x_i, t|x_i', t')$

$$\bar{G}(y, t|y_i', t'; x', z'; M, N) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x_i, t|x_i', t')\exp(iMx + iNz)\,dx\,dz \tag{3.1}$$

and its inverse is

$$G(x_i, t|x_i', t') = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{G}(y, t|y', t'; x', z'; M, N)\exp[-i(Mx + Nz)]\,dM\,dN \tag{3.2}$$

where $i = (-1)^{1/2}$. By the usual procedure for the construction of Green's functions by Fourier Transform [16], we obtain

$$\frac{\partial^2 \bar{G}}{\partial y^2} - 2i(Mv_{12} + Nv_{23})\frac{\partial \bar{G}}{\partial y} - (M^2 v_{11} + N^2 v_{33} + 2MNv_{13})\bar{G} - \frac{1}{\alpha_2}\frac{\partial \bar{G}}{\partial t}$$
$$= -\frac{1}{\alpha_2 w(x_i)}\exp[i(Mx' + Nz')\delta(t - t')\delta(y - y'). \tag{3.3}$$

We seek the solution of (3.3) in the form

$$\bar{G}(y, t|y', t', x', z', M, N) = \Psi(y|y'; M, N)\exp[i(Mx' + Nz') - \alpha_2\lambda^2(t - t')]. \tag{3.4}$$

We substitute (3.4) into (3.3) to obtain

$$\Psi'' - 2i(Mv_{12} + Nv_{23})\Psi' + (\lambda^2 - M^2 v_{11} - N^2 v_{33} - 2MNv_{13})\Psi = -\frac{1}{w(y)}\delta(y - y') \tag{3.5}$$

where the superscript primes designate differentiations with respect to y. This equation can be reduced to self-adjoint form with the weight function $w(y) = \exp[i2(Mv_{12} + Nv_{23})y]$. Therefore, we obtain

$$\Psi(y/y') = Y(y|y')\exp[i(Mv_{12} + Nv_{23})(y - y')]$$

where $Y(y|y')$ satisfies

$$Y''(y) + (\lambda^2 - M^2\beta_{11}^2 - N^2\beta_{33}^2 - 2MN\gamma^2)Y(y) = -\delta(y - y'). \tag{3.6}$$

Equation (3.6) can be easily solved for a specific region with a given type of boundary condition by standard methods [16]. Once $Y(yy')$ is found, the Green's function is given by

$$G(x_i, t|x_i', t') = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Y(y|y')\exp\{-iM[(x - x') - v_{12}(y - y')]$$
$$-iN[(z - z') - v_{23}(y - y')] - \alpha_2\lambda^2(t - t')\}\,dM\,dN. \tag{3.7}$$

In principle, the integrals in (3.7) can be evaluated by residue calculus. Due to the coupling of M and N in (3.6), however, it is inconvenient to evaluate the integrals except for cases where $v_{12}v_{23} = v_{13}$ or the problem is independent of z. Since M is arbitrary, we may let $M = P - AN$ where P is another arbitrary constant and A a constant yet to be determined. Replacing M in (3.6) by $P - AN$, we find that if $A = \gamma/\beta_{11}^2$ then (3.6) takes the form:

$$Y''(y) + [\lambda^2 - M^2\beta_{11}^2 - N^2\sigma^2]Y(y) = -\delta(y - y') \tag{3.8}$$

where P has been rewritten as M since either one is arbitrary. Finally, we obtain the following general formula for Green's functions associated with all the problems concerned in this paper:

$$G(x_i, t|x_i', t') = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Y(y|y')\exp\{-iM[(x-x')-v_{12}(y-y')] -iN\left[(z-z')-\varepsilon(y-y')-\frac{\gamma}{\beta_{11}^2}(x-x')\right]-\alpha_2\lambda^2(t-t')\}\,dM\,dN. \quad (3.9)$$

Now the problem is to investigate how $G(x_i, tx_i', t')$ will satisfy the boundary conditions, for the half space,

$$b\frac{\partial G}{\partial n^+}+hG = 0 \text{ at } y = 0; \quad G = 0 \text{ at } y = \infty \quad (3.10)$$

and, for the slab,

$$b\frac{\partial G}{\partial n^+}+hG = 0 \text{ at } y = 0 \text{ and } L. \quad (3.11)$$

Substituting (3.7) or (3.9) into (3.10) and (3.11) and assuming that the integration and differentiation can be interchanged, we readily find that if $Y(y|y')$ satisfies the boundary conditions for the half space

$$bY'-hY = 0 \text{ at } y = 0; \quad Y = 0 \text{ at } y = \infty \quad (3.12)$$

and for the slab

$$bY'-hY = 0 \text{ at } y = 0; \quad bY'+hY = 0 \text{ at } y = L \quad (3.13)$$

then $G(x_i, tx_i', t')$ given by (3.7) or (3.9) satisfies (3.10) and (3.11).

The Green's functions for steady problems can be obtained by two methods: (i) integrating $G(x_i, t|x_i', 0)$ with respect to t from $t = 0$ to $t = \infty$, and multiplying the result by α_2:

$$G(x_i|x_i') = \alpha_2\int_0^{\infty} G(x_i, t|x_i', t', 0)\,dt \quad (3.14)$$

and (ii) solving the governing equations of $G(x_i|x_i')$ by the same way as in obtaining the Green's function for unsteady state. The result is

$$G(x_i|x_i') = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Y(y|y')\exp\left\{-iM[(x-x')-v_{12}(y-y')] -iN\left[(z-z')-\varepsilon(y-y')-\frac{\gamma}{\beta_{11}^2}(x-x')\right]\right\}dM\,dN \quad (3.15)$$

where $Y(y|y)$ satisfies

$$Y''-[M^2\beta_{11}^2+N^2\sigma^2]Y = -\delta(y-y'). \quad (3.16)$$

Again, if Y satisfies boundary conditions (3.12) or (3.13) then $G(x_i|x_i')$ given by (3.15) satisfies conditions (3.10) and (3.11).

It is clearly seen that when we have solved the anisotropic problems by this formula, we have also solved the corresponding isotropic and orthotropic problems by setting $v_{ij} = 0$ and $v_{ii} = 1$ and $v_{ij} = 0$, respectively, as well as problems of any system of anisotropy, such as monoclinic, orthohomic, tetragonal, etc. [1, 2, 15]. It may be pointed out here that most of the integrals in this paper can be evaluated by residue calculus, and many of them can be found in mathematical manuals [24, 25].

4. FREE SPACE

Green's functions associated with unsteady and steady problems in an infinite anisotropic region were first reported by Levi and known as Levi's parametric functions, or fundamental solutions, which are well known in theory of integral and differential equations [21, 22]. Although Levi's functions are Green's functions, they are not symmetrical with respect to any plane of x_j = constant. Therefore, their usefulness in applied mathematics is limited to the numerical solution of problems in arbitrary regions [11, 12] or to the analytical solution of steady problems in circular or elliptic plane regions with anisotropy homogeneous in rectangular coordinates [3]. The function $Y(y|y')$ governed by (3.8) and (3.16), however, can be symmetrical with any y-plane. It is therefore advantageous to obtain the Green's functions in infinite region by formulae (3.9) and (3.14). The solution of (3.8) for $Y(yy')$ satisfying $Y = 0$ for $y = \pm\infty$ may be taken in the form:

$$\frac{1}{2\pi}\exp(-P|y-y'|) \quad (4.1)$$

where

$$P^2 = -(\lambda^2 - M^2\beta_{11}^2 - N^2\sigma^2) \tag{4.2}$$

or

$$\frac{1}{2\pi}\exp[-iP(y-y')] \tag{4.1'}$$

where

$$P^2 = \lambda^2 - M^2\beta_{11}^2 - N^2\sigma^2. \tag{4.2'}$$

Applying (4.1) or (4.1′) to (3.9) we obtain

$$G(x_i, t|x_i', t') = \frac{1}{2[\pi\alpha_2(t-t')]^{1/2}}\exp\left[-\frac{(y-y')^2}{4\alpha_2(t-t')}\right]g(x_i, t|x_i', t') \tag{4.3}$$

where

$$g(x_i, t|x_i', t') = \frac{1}{4\pi\alpha_2\beta_{11}(t-t')\sigma}\exp[-R_1^2/4\alpha_2(t-t')] \tag{4.4}$$

and

$$R_1^2 = \frac{1}{\beta_{11}^2}[(x-x') - v_{12}(y-y')]^2 + \frac{1}{\sigma^2}[(z-z') - \varepsilon(y-y') - (\gamma/\beta_{11}^2)(x-x')]^2. \tag{4.5}$$

Applying (4.3) to formula (3.14) gives the Green's function for steady state

$$G(x_i x_i') = \frac{1}{4\pi\beta_{11}\sigma}\frac{1}{R} \tag{4.6}$$

where $R^2 = R_1^2 + (y-y')^2$.

If $G(x_i|x_i')$ is independent of z, we can obtain $G(x, y|x', y')$ from formula (3.14), or (3.15) through the solution of (3.16) for $Y(yy')$ satisfying $Y = 0$ for $y = \pm\infty$, or by integrating (4.6) with respect to z over $(-\infty, \infty)$ to obtain

$$G(x, y|x', y') = \frac{1}{2\pi\beta_{11}}\ln\frac{1}{R_2} \tag{4.7}$$

where

$$R_2^2 = \frac{1}{\beta_{11}^2}[(x-x') - v_{12}(y-y')]^2 + (y-y')^2. \tag{4.8}$$

5. HALF SPACE

For the half space, the Green's functions with boundary conditions of the first and second kinds can be readily written down by the method of image as follows:

$$G(x_i, t|x_i', t') = \frac{1}{2[\pi\alpha_2(t-t')]^{1/2}}\exp[-(y-y')^2/4\alpha_2(t-t')] \pm \exp[-(y+y')^2/4\alpha_2(t-t')\cdot g(x_i, t|x_i', t')] \tag{5.1}$$

where the minus and plus signs are for boundary conditions of the first and second kinds respectively; and $g(x_i, t|x_i', t')$ is given in (4.4). For the boundary condition of the third kind, the solution (3.8) is

$$Y(y|y') = \frac{h}{2\pi}\{\exp[-P(y+y')] + \exp[-P(y-y')]\} - \frac{h}{\pi(P+h)}\exp[-P(y+y')] \tag{5.2}$$

where P^2 is given by (4.2). Substituting (5.2) into (3.9) and performing the integrations gives

$$G(x_i, t|x_i't') = \left[\left\{\exp\left(-\frac{(y-y')^2}{4\alpha_2(t-t')}\right) + \exp\left(-\frac{(y+y')^2}{4\alpha_2(t-t')}\right)\right\}\frac{1}{2[\pi\alpha_2(t-t')]^{1/2}}\right.$$
$$\left. - 2h[\pi\alpha_2(t-t')]^{1/2}\exp[\alpha_2 h^2(t-t') + h(y+y')]\,\mathrm{erfc}\left(\frac{y+y'}{2[\alpha_2(t-t')]^{1/2}} + h[\alpha_2(t-t')]^{1/2}\right)\right]g(x_i, t|x_i', t') \tag{5.3}$$

where erfc is the complementary error function and $g(x_i, t|x_i', t')$ is given in (4.4).

The Green's functions for steady Dirichlet and Neumann problems can be similarly found by the method of image as follows:

$$G(x_i|x_i') = \frac{1}{4\pi\beta_{11}\sigma}\left(\frac{1}{R} \pm \frac{1}{R_3}\right) \tag{5.4}$$

where the minus and plus signs are for Dirichlet and Neumann problems, respectively; and

$$\left.\begin{aligned} R^2 &= R_1^2+(y-y')^2 \\ R_3^2 &= R_1^2+(y+y')^2 \end{aligned}\right| . \tag{5.5}$$

The Green's function for the three-dimensional problem in steady state with the boundary condition of third kind can be obtained by substituting (5.3) into (3.14):

$$G(x_0|x_i) = \frac{1}{4\pi}\left(\frac{1}{R}+\frac{1}{R_3}-2h\int_{y'}^{\infty} e^{-h(\xi-y')}\frac{d\xi}{R_4}\right) \tag{5.6}$$

where

$$R_4^2 = R_1^2+(y+\xi)^2. \tag{5.7}$$

Green's functions for two-dimensional cases, $G(x, y, t|x', y', t')$ and $G(x, y|x', y')$ can be obtained by the same way. Those in unsteady state are in the same forms as (5.1) and (5.3) provided $g(x_i, t|x_i', t')$ is replaced by $g_1(x, y, t|x', y', t')$ defined by:

$$g_1(x, y, t|x', y', t') = \frac{1}{2[\pi\alpha_2\beta_1(t-t')]^{1/2}}\exp\{-[(x-x')-v_{12}(y-y')]^2/4\alpha_2\beta_{11}^2(t-t')\} \tag{5.8}$$

and those in steady state with boundary conditions of the first and second kinds are

$$G(x, y|x', y') = \frac{1}{2\pi\beta_{11}}\left(\ln\frac{1}{r_1}\pm\ln\frac{1}{r_2}\right) \tag{5.9}$$

where

$$\begin{aligned} r_1^2 &= (y-y')^2+\frac{1}{\beta_{11}^2}[(x-x')-v_{12}(y-y')]^2 \\ r_2^2 &= (y+y')^1+\frac{1}{\beta_{11}^2}[(x-x')-v_{12}(y-y')]^2 \end{aligned} \tag{5.10}$$

and the plus and minus signs have the same meaning as in (5.4). For the boundary condition of third kind we may again use (3.14) to obtain

$$G(x, y|x', y') = \frac{1}{2\pi\beta_{11}}\left[\ln\frac{1}{r_1}+\ln\frac{1}{r_2}-2h\int_{y'}^{\infty} e^{-h(\xi-y')/k}\left(\ln\frac{1}{r_4}\right)d\xi\right] \tag{5.11}$$

where

$$r_4^2 = (y+\xi)^2+[(x-x')-v_{12}(y-y')]^2/\beta_{11}^2.$$

6. INFINITE SLAB

Consider first unsteady problems. The solutions of (3.8) satisfying (3.13) for boundary conditions of the first, second and third kinds are respectively,

$$Y_{1m}(y|y') = \frac{2}{L}\sin\frac{m\pi y}{L}\sin\frac{m\pi y'}{L} \qquad m = 1, 2, 3, \ldots, \infty \tag{6.1}$$

$$Y_{2m}(y|y') = \frac{2}{L}\cos\frac{m\pi y}{L}\cos\frac{m\pi y'}{L} \qquad m = 1, 2, 3, \ldots, \infty \tag{6.2}$$

$$Y_{3m}(y|y') = \frac{2}{L}\left(\frac{(\omega_m\cos\omega_m y+h\sin\omega_m y)(\omega_m\cos\omega_m y'+h\sin\omega_m y')}{\omega_m^2+h^2+2h/L}\right) \tag{6.3}$$

where ω_m are the roots of the transcendental equation

$$\tan\omega L = \frac{2h\omega}{\omega^2-h^2} \tag{6.4}$$

and

$$\lambda^2 = \left(\frac{m\pi}{L}\right)^2+M^2\beta_{11}^2+N^2\sigma^2$$

for boundary conditions of the first and second kinds, and

$$\lambda^2 = \omega_m^2+M^2\beta_{11}^2+N^2\sigma^2$$

for the third-kind boundary condition. Substituting (6.1), (6.2) and (6.3) into (3.9), performing the integrations, and noting that $m = 0$ is also an eigenvalue for the Neumann problem, we obtain the Green's functions with boundary conditions of the first, second and third kinds, respectively, as follows:

$$G(x_i, t|x_i', t') = \sum_{m=1}^{\infty} Y_{1m}(y|y')\exp\left[-\alpha_2\left(\frac{m\pi}{L}\right)^2(t-t')\right]\cdot g(x_i, t|x_i', t') \tag{6.5}$$

$$G(x_i, t|x_i', t') = \left[\frac{1}{L}-\sum_{m=1}^{\infty} Y_{2m}(y|y')\exp\left\{-\alpha_2\left(\frac{m\pi}{L}\right)^2(t-t')\right\}\right]g(x_i, t|x_i', t') \tag{6.6}$$

$$G(x_i, t|x_i', t') = \sum_{m=1}^{\infty} Y_{3m}(y|y')\exp[-\alpha_2\omega_m^2(t-t')]g(x_i, t|x_i', t') \tag{6.7}$$

where $g(x_i, t|x'_i, t')$ is given in (4.4). For steady state, we apply (6.5) and (6.7) to (3.14) to obtain the Green's functions with boundary conditions of the first and third kinds, respectively,

$$G(x_i|x'_i) = \frac{1}{2\pi\beta_{11}\sigma}\sum_{m=1}^{\infty} Y_{1m}(y|y')K_0\left(\frac{m\pi}{L}R_1\right) \tag{6.8}$$

$$G(x_i|x'_i) = \frac{1}{2\pi\beta_{11}\sigma}\sum_{m=1}^{\infty} Y_{3m}(y|y')K_0(\omega_m R_1) \tag{6.9}$$

where values of ω_m are given by (6.4); R_1 is given in (4.5); and K_0 is the modified Bessel function of the second kind and order zero. If we substitute (6.6) into (3.14), we shall see that the Green's function for the Neumann problem does not exist, and we have to construct the generalized Green's function which will not be shown here.

For an infinite strip in the region, $0 < y < L$, $x < \infty$, with boundary conditions of the first kind, the solution of (3.16) for $y < y'$ is

$$Y(y|y') = \frac{\exp[M\beta_{11}(y-y')] - \exp[-M\beta_{11}(y+y')]}{2M\beta_{11}[1-\exp(-2M\beta_{11}L)]} + \frac{\exp[M\beta_{11}(y+y')] - \exp[-M\beta_{11}(y-y')]}{2M\beta_{11}[1-\exp(2M\beta_{11}L)]}. \tag{6.10}$$

and, for $y > y'$, we simply interchange y and y'. Substituting (6.10) into (3.15) and evaluating the integral, we obtain the Green's function for the two-dimensional Dirichlet problem

$$G(x, y|x', y') = \frac{1}{\pi\beta_{11}}\sum_{m=1}^{\infty}\frac{1}{m}\sin\frac{m\pi y}{L}\sin\frac{m\pi y'}{L}\exp\left[-\frac{m\pi}{\beta_{11}L}|(x-x')-v_{12}(y-y')|\right] \tag{6.11}$$

or

$$G(x, y|x', y') = \frac{1}{4\pi\beta_{11}}\ln\frac{\sin^2\left[\frac{\pi}{2L}(y+y')\right] + \sinh^2\left\{\frac{\pi}{2\beta_{11}L}[(x-x')-v_{12}(y-y')]\right\}}{\sin^2\left[\frac{\pi}{2L}(y-y')\right] + \sinh^2\left\{\frac{\pi}{2\beta_{11}L}[(x-x')-v_{12}(y-y')]\right\}}. \tag{6.12}$$

The Green's function for the two-dimensional, steady problem with third kind boundary conditions can be obtained by the same way as

$$G(x, y|x', y') = \frac{1}{\beta_{11}L}\sum_{m=1}^{\infty}\frac{(\omega_m\cos\omega_m y + h_2\sin\omega_m y)(\omega_m\cos\omega_m y' + h_2\sin\omega_m y)}{\omega_m(\omega_m^2+h_2^2+2h_2/L)} \times \exp[-(\omega_m/\beta_{11})|(x-x')-v_{12}(y-y')|]. \tag{6.13}$$

It is to be remarked that if formulae (3.15) and (3.16) were used, closed-form solutions of (6.8), (6.9) and (6.13) are difficult to obtain by the residue theorem.

7. ANISITROPIC PARAMETERS AND AN ILLUSTRATIVE EXAMPLE

From the above results of Green's functions it is seen that the material properties and heat-transfer coefficient are grouped into the following parameters:

$$\alpha_2 t/L^2, \quad h/L \tag{7.1}$$

$$v_{12}, \quad \beta_{11}, \quad \gamma = v_{13} - v_{12}v_{23},$$

$$\varepsilon = v_{23} - v_{12}\frac{\gamma}{\beta_{11}^2}, \quad \sigma^2 = \beta_{33}^2 - \gamma^2/\beta_{11}^2. \tag{7.2}$$

If the governing equation is divided by k_{11} of k_{33} similar sets of parameters as in (7.1) and (7.2) can be obtained. The parameters in (7.1) are well known as the Fourier and Biot moduli in isotropic problems and therefore, discussion on them is not needed. The parameters in (7.2) are of great importance. According to irreversible thermodynamics, as mentioned earlier, $\beta_{ii} > 0$ and $v_{ij}(i \neq j)$ can be positive or negative. From the definitions of γ and ε, they can also be either positive or negative. From the definition of σ and the expression of Green's functions, σ must be larger than zero and the Green's function does not exist if $\beta_{11} = 0$ or $\sigma = 0$. The findings are of great importance in the investigation of anisotropic problems and have not been reported by previous investigators.

In order to gain some specific insights into the anisotropic effects to the heat transfer, a simple example may be considered: heat conduction in the infinite strip $0 \leqslant y \leqslant L$ and $x < \infty$, in steady state, without heat generation, and with boundary conditions:

$$T(x, 0) = f_1(x) \qquad T(x, L) = f_2(x). \tag{7.3}$$

The Green's function associated with this problem is given in two forms, one in closed form (6.12), and the other in series (6.11). Applying (2.6) to the present problem, we obtain

$$T(x, y) = -\int_{-\infty}^{\infty} f_1(x')\frac{\partial}{\partial n^+}G(x, y|x', 0)\,dx' - \int_{-\infty}^{\infty} f_2(x')\frac{\partial}{\partial n^+}(x, y|x', L)\,dx'. \tag{7.4}$$

If (6.11) is used for $G(x, y|x', y')$, the transverse derivatives are obtained:

$$\frac{\partial}{\partial n^+} G(x, y|x', 0) = -\frac{1}{\beta_{11} L} \sum_{m=1}^{\infty} \sin\frac{m\pi y}{L} \exp[-m\pi|(x-x') - \nu_{12} y|/(\beta_{11} L)] \tag{7.5}$$

$$\frac{\partial}{\partial n^+} G(x, y|x', L) = \frac{1}{\beta_{11} L} \sum_{m=1}^{\infty} (-1)^n \sin\frac{m\pi y}{L} \exp[-m\pi|(x-x') - \nu_{12}(y-L)|/(\beta_{11} L)]. \tag{7.6}$$

If (7.5) and (7.6) are substituted into (7.4), the integrations can be easily performed and expressed in terms of tabulated functions if $f_1(x)$ and $f_2(x)$ are elementary functions. The resulted series converge slowly near boundaries.

If the Green's function in the form (6.12) is used, there results

$$\frac{\partial}{\partial n^+} G(x, y|x', 0) = -\frac{1}{2\beta_{11} L} \frac{\sin\frac{\pi y}{L}}{\cosh\frac{\pi[(x-x') - \nu_{12} y]}{\beta_{11} L} - \cos\frac{\pi y}{L}} \tag{7.7}$$

$$\frac{\partial}{\partial n^+} G(x, y|x', L) = \frac{1}{2\beta_{11} L} \frac{\sin\frac{\pi(y-L)}{L}}{\cosh\frac{\pi[(x-x') - \nu_{12}(y-L)]}{\beta_{11} L} - \cos\frac{\pi(y-L)}{L}}. \tag{7.8}$$

Substituting (7.7) and (7.8) into (7.4) gives $T(x, y)$, which is identical with that reported by Tauchert and Akoz [6], except in notation. To show that the results (7.5) and (7.6) are equivalent to (7.7) and (7.8), the former may be considered as the imaginery parts of the function

$$[\exp(i\pi y/L) \exp\{-\pi|(x-x') - \nu_{12} \bar{y}|/(\beta_{11} L)\}]^m$$

where $\bar{y} = y$ for (7.5) and $\bar{y} = y - L$ for (7.6). A little further deduction gives the results of (7.7) and (7.8).

From the solution of $T(x, y)$, the effects of anisotropy to the temperature and heat flow can be easily discussed. For instance, consider the case:

$$\begin{aligned} f_1(x) &= 0, \quad \text{for} \quad |x| < \infty; \\ f_2(x) &= 0, \quad \text{for} \quad 0 > x > 2L \\ f_2(x) &= f_2(-x) \quad \text{for} \quad 0 < x < 2L. \end{aligned} \tag{7.9}$$

If the medium is isotropic or orthotropic, the temperature distribution and heat flow lines are symmetric with respect to the plane $x = L$. In an anisotropic medium, however, no such symmetry exists. If ν_{12} is positive, the temperature is higher in the left hand side of $x = L_1$ than in the right hand side at the same corresponding points with respect to $x = L$. Some calculated results of isotherms for $\beta_{11} = 7/8$, $\nu_{12} = 1/4$ and $f_2(x) = T_0 \sin(\pi x/2L)$ are shown in Fig. 1. The heat flow lines are depicted in Fig. 2(b) for positive ν_{12} and in Fig. 3(b) for negative ν_{12}. However, patterns of isothermal and heat flow lines are the same if they are looked from the face of the paper for one case and from the backside of the paper for the other. From physical points of view, for positive ν_{12} the anisotropy of the material may be depicted as in Fig. 2(a) with heat flow lines in Fig. 2(b), and for negative ν_{12}, they may be depicted as in Figs. 3(a) and (b). These discussions are qualitatively in agreement with those reported in [11] for problems in finite regions.

Though the material anisotropy is represented by ν_{ij}, yet its effects to the temperature distribution and hence to heat transfer depend on the parameters in (7.2). It can be clearly seen from the definitions of γ and ε, that for three-dimensional problems, the cross-conductivity coefficients ν_{13} and ν_{23} affect the heat transfer only through γ and ε. In other words, a strong anisotropy of the material in xz and/or yz plane (i.e. ν_{13} and ν_{23} are large)

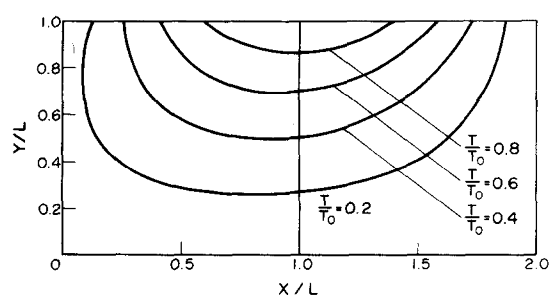
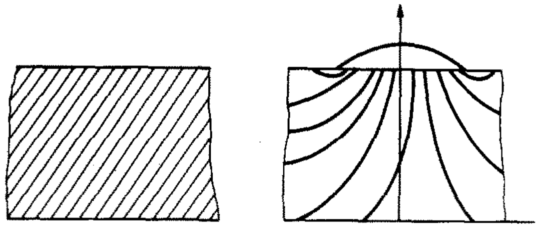
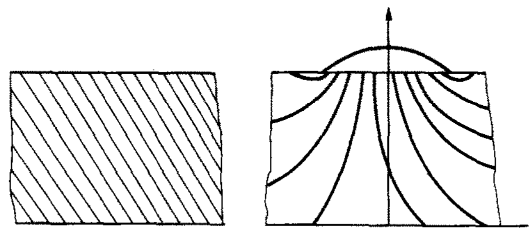


FIG. 1. Temperature distribution in $0 \leqslant x \leqslant 2L$, $0 \leqslant y \leqslant L$ of a slab.



2(a) 2(b)

FIG. 2. (a) Orientation of anisotropy for positive ν_{12}.
(b) Heat-flow lines for positive ν_{12}.



3(a) 3(b)

FIG. 3. (a) Orientation of anisotropy for negative ν_{12}.
(b) Heat-flow lines for negative ν_{12}.

may have small effects to the heat transfer if γ and ε are small. The parameters of β_{11} and σ play the role of scale factors to the spatial coordinates, as well as to the magnitude of temperature. From the results of Green's functions for three dimensional problems, it is seen that ε and γ/β_{11}^2 play similar roles as ν_{12}. Therefore their effects to the temperature field can be discussed in the like manner as those discussed above on ν_{12}.

8. CONCLUDING REMARKS

It can be shown without difficulty that the usually used transformation of coordinates into principal ones is not useful for problems in a bounded or partially bounded region, because the domain will be deformed and the boundary condition or conditions, particularly those of the second and third kinds, will become more complicated. However, from the success of reducing (3.6) to the form of (3.8) as well as from the results of Green's functions, we can readily see that the differential equation (2.1) and conditions, (2.3) and (2.4) can be transformed into the same forms for isotropic media, i.e.

$$\frac{\partial^2 T}{\partial \xi_i^2} - \frac{1}{\alpha^2}\frac{\partial T}{\partial t} + Q(\xi_i, t) = 0 \qquad \text{in } \Omega(\xi_i),\ t > 0 \tag{8.1}$$

$$b\frac{\partial T}{\partial \xi_2} = hT = f(\xi_i) \qquad \text{on } S(\xi_s),\ t > 0 \tag{8.2}$$

$$T = F(\xi_i) \qquad \text{in } \Omega(\xi_i),\ t = 0 \tag{8.3}$$

where

$$\xi_1 = \frac{1}{\beta_{11}}(x_1 - \nu_{12}x_2), \qquad \xi_2 = x_2$$

$$\xi_3 = \frac{1}{\sigma}\left(x_3 - \varepsilon x_2 - \frac{\gamma}{\beta_{11}^2}x_1\right). \tag{8.4}$$

Clearly, however, the transformation of coordinates defined by (8.4) is useful for problems of the first class with anisotropy homogeneous in rectangular coordinates. It fails for first-class problems in other coordinate systems and for problems of the second and third classes in any coordinate system.

The anisotropic parameters shown in (7.2) are the most significant and useful ones. Although each of ν_{ij} represents a directional anisotropy of a material, yet their effects to the heat transfer are mutually related and one may oppose the other. This is of great importance in the production of a specially-purposed material in which the heat transfer in one direction may be many times larger than in other directions.

If the solution of a problem involves an infinite series, the series usually converges as fast as those of the corresponding isotropic problem.

All problems in this paper can be solved directly by Fourier transform provided the technique in obtaining (3.8) is employed. For problems of other classes, any integral transform will be no longer useful because of the impossibility of finding a kernel which can satisfy the homogeneous boundary conditions. It may be just mentioned here that Green's function can be constructed for those problems according to the Green's functions for first-class problems. We hope to report such results elsewhere.

REFERENCES

1. W. A. Wooster, *A Textbook in Crystal Physics*. University Press, Cambridge (1938).
2. J. Y. Nye, *Physical Properties of Crystals*. Clarendon Press, Oxford (1960).
3. V. D. Kuprudze, *Potential Methods in the Theory of Elasticity*, Chapter 9. Israel Programs for Scientific Trans., Haifa (1965).
4. S. G. Lekhnitskii, *Theory of Elasticity of an Anisotropic Elastic Body*, Translated by P. Fern. Holden Day, San Francisco (1963).
5. D. L. Clements, Thermal stress in an anisotropic elastic half-space, *SIAM J. Appl. Math.* **24**(1), 332–337 (1973).
6. T. R. Tauchert and A. Y. Akoz, Stationary temperature and stress fields in anisotropic elastic slab, *J. Appl. Mech.* 647–650 (1975).
7. D. Turkan and O. H. Tuna, A heat conduction theory of layered composites, *METU J. Pure Appl. Sci.* **8**, 69–93 (1975).
8. J. Padovan, Solution of transient temperature fields in laminated anisotropic slabs and cylinders, *Int. J. Engng Sci.* **13**, 247–260 (1973).
9. G. Horvay, R. Mani, M. A. Veluswami and G. E. Zinsmeister, Transient heat conduction in laminated composites, *J. Heat Transfer* **95**, 309–317 (1973).
10. K. Katayama *et al.*, Transient heat conduction in anisotropic solids, *Proc. Int. Conf. Heat Mass Transfer, Tokyo, Japan*, Cul. 4, pp. 137–141 (1975).
11. Y. P. Chang, C. S. Kang and D. J. Chen, The use of fundamental Green's functions for the solution of problems of heat conduction in anisotropic media, *Int. J. Heat Mass Transfer* **16**, 1905–1918 (1973).
12. D. J. Chen, The use of Green's functions for the solution of problems of heat conduction in irregularly-shaped anisotropic solids, Ph.D. Dissertation, Dept. Mech. Engng. State University of New York at Buffalo (1973).
13. M. H. Cobble, Nonlinear anisotropic temperature distribution in a wedge, *Int. J. Heat Mass Transfer* **17**, 379–380 (1974).
14. Y. P. Chang and C. H. Tsou, Heat conduction in an anisotropic medium homogeneous in cylindrical coordinates—unsteady state, *J. Heat Transfer* **99**, 41–47 (1977).
15. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, Clarendon Press, Oxford (1960).
16. I. Stakgold, *Boundary Value Problems of Mathematical Physics*, Vol. II. MacMillan, New York (1968).
17. S. L. Sobolev, *Partial Differential Equations of Mathematical Physics*. Pergamon Press, Oxford (1964).
18. Y. P. Chang and C. H. Tsou, Heat conduction in an anisotropic medium, homogeneous in cylindrical coordinates, steady state, *J. Heat Transfer* **99**, 132–134 (1977).
19. I. N. Sneddon, *Fourier Transforms*. McGraw-Hill, New York (1951).
20. I. Prigogine, *Introduction to Thermodynamics of Irreversible Processes*. Charles Thomas, Springfield, Ill. (1955).
21. A. Friedman, *Partial Differential Equations of Parabolic Type*. Prentice Hall, Englewood Cliffs, N.J. (1964).
22. W. Pogorzeiski, *Integral Equations and Their Applications*. Pergamon Press, Oxford (1966).
23. C. Lanzos, *Linear Differential Operators*. Van Nostrand, New York (1961).
24. A. Erdelyi *et al.*, *Tables of Integral Transforms*. McGraw-Hill, New York (1955).
25. I. S. Gradshteyn *et al.*, *Tables of Integrals, Series and Products*. Academic Press, New York (1965).

SOLUTION ANALYTIQUE DE LA CONDUCTION THERMIQUE DANS UN MATERIAU ANISOTROPE ET DES REGIONS INFINIES, SEMI-INFINIES OU LIMITEES PAR DEUX PLANS

Résumé—Cet article est l'un d'une série publiée dans la littérature et concernant la solution analytique de la conduction thermique dans les milieux anisotropes. Compte tenu des difficultés mathématiques et des méthodes de résolution, les problèmes sont divisés en trois classes. Ici seuls des problèmes de la première classe, avec des matériaux homogènes et anisotropes en coordonnées rectangulaires, sont résolus à l'aide des fonctions de Green. On donne la solution d'un problème spécifique et on discute les effets de l'anisotropie sur le champ de température et sur le flux de chaleur, du point de vue mathématique aussi bien que physique.

ANALYTISCHE LÖSUNG FÜR DIE WÄRMELEITUNG IN ANISOTROPEN, UNENDLICHEN UND HALBUNENDLICHEN KÖRPERN SOWIE IN UNENDLICH AUSGEDEHNTEN PLATTEN

Zusammenfassung—Diese Arbeit gehört zu einer Serie von Veröffentlichungen über die analytische Lösung von Wärmeleitproblemen in anisotropen Körpern. Entsprechend der mathematischen Schwierigkeiten und der Lösungsmethoden werden die Probleme in drei Klassen eingeteilt. In der vorliegenden Arbeit werden die Probleme der ersten Klasse mit homogener Anisotropie in rechtwinkligen Koordinaten unter Verwendung der Greenschen Funktionen gelöst. Die Lösungsmethode wird auf ein spezielles Problem angewandt, und es werden die Einflüsse der Anisotropie auf das Temperaturfeld und den Wärmestrom aus mathematischer und physikalischer Sicht diskutiert.

АНАЛИТИЧЕСКОЕ ИССЛЕДОВАНИЕ ТЕПЛОПРОВОДНОСТИ В АНИЗОТРОПНЫХ СРЕДАХ В БЕСКОНЕЧНОЙ И ПОЛУБЕСКОНЕЧНОЙ ОБЛАСТЯХ И В ОБЛАСТИ, ОГРАНИЧЕННОЙ ДВУМЯ ПЛАСТИНАМИ

Аннотация — Настоящая статья относится к ряду работ по аналитическому решению задач теплопроводности в анизотропных средах, представленных автором для опубликования в печати. По методам и сложности решения задачи подразделяются на 3 класса. В настоящей работе рассматриваются задачи 1-го класса для анизотропных однородных сред в прямоугольных координатах, решаемые с помощью функции Грина. Приведено решение частной задачи. Рассматривается влияние анизотропности на температурное поле и величину теплового потока как с математической, так и с физической точек зрения.